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The non-existence of a regular exceptional family of elements. A necessary and sufficient condition. Applications to complementarity theory

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Abstract This paper is the second part of our recent work [Isac and Németh, J Optim Theory Appl (forthcoming)]. Our goal is now to present some new results related to the non-existence of a regular exceptional family of elements (REFE) for a mapping and to show how can they be applied to complementarity theory.

Keywords Non-existence of a regular exceptional family of elements · Complementarity problems

1 Introduction

This paper is a continuation of our recent paper [10] in which we considered the notion of regular exceptional family of elements (REFE) and we defined the class of REFE-acceptable mappings, related to complementarity problems. A mapping f is called a REFE-acceptable mapping with respect to a closed convex cone K in a Hilbert space H, if the non-existence of a regular exceptional family of elements implies that the nonlinear complementarity problem $NCP(f, K)$ defined by f and K has a solution. Several classes of REFE-acceptable mappings were given. A main result in [10] is a necessary and sufficient condition for the non-existence of regular exceptional families of elements. In this paper we present a few geometrical variants of this interesting result, and we establish some relations between the necessary and sufficient conditions and the eigenvectors of the mapping. In this way we obtain some existence theorems for nonlinear or linear complementarity problems.

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2 Preliminaries

We denote by $(H, \langle \cdot, \cdot \rangle)$ a Hilbert space and by $K \subset H$ a closed convex cone in H, that is a closed subset of H such that

1. $K + K \subseteq K$,
2. $\lambda K \subseteq K$, for all $\lambda \in \mathbb{R}_+$.

If the cone K also satisfies the property $K \cap (-K) = \{0\}$ we say that K is a pointed convex cone. If K is given, $x \le y$ if and only if $y - x \in K$. The *dual cone* K^* of K is the closed convex cone defined by

$$K^* = \{y \in H \mid \langle x, y \rangle \ge 0 \text{ for all } x \in K\}.$$

Here $\ge$ is the inequality between real numbers and should not be confounded with the vectorial relation defined by the cone. If $D \subset H$ is a closed convex set, we denote by P_D the projection onto D, that is the mapping $P_D\colon H \to D$ defined for every $x \in H$ by: $P_D(x)$ is the unique element in D such that $\|x - P_D(x)\| \le \|x - y\|$ for any $y \in D$. If $f\colon H \to H$ is a mapping, the Nonlinear Complementarity Problem defined by f and the cone K is

$$NCP(f, K)\colon \begin{cases} \text{find } x^* \in K \text{ such that} \\ f(x^*) \in K^* \text{ and } \langle x^*, f(x^*) \rangle = 0 \end{cases}.$$

If f has the form $f(x) = Ax + b$, where A is a linear continuous mapping from H into H and b is an element in H, the problem $NCP(f, K)$ is called the Linear Complementarity Problem, defined by A, b and the cone K and it is denoted by $LCP(A, b, K)$. The problem $NCP(f, K)$ is the model for many problems considered in Optimization, Economics, Mechanics and Engineering [1–3,5,6,11]. Generally, the complementarity problems are related to equilibrium problems considered in Physics and Economics.

3 REFE-acceptable mappings

The notion of REFE-acceptable mapping was defined in [10]. For defining this notion we need the definition of a regular exceptional family of elements due to Isac et al. [10].

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f\colon H \to H$ a mapping. We say that a family of elements $\{x_r\}_{r>0} \subset K$ is a *regular exceptional family of elements* (shortly REFE) for f with respect to K, if for every real number $r > 0$, there exists a real number $\mu_r > 0$ such that the vector $u_r = \mu_r x_r + f(x_r)$ satisfies the following conditions:

1. $u_r \in K^*$,
2. $\langle u_r, x_r \rangle = 0$,
3. $\|x_r\| = r$.

Definition 3.1 We say that a mapping $f : H \to H$ is a *REFE-acceptable* mapping with respect to a closed convex cone $K \subset H$ if either the problem $NCP(f, K)$ has a solution, or the mapping f has a REFE with respect to K.

From Definition 3.1 we deduce the following result.

Corollary 3.1 *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f\colon H \to H$ a REFE-acceptable mapping with respect to K. If f is without a REFE with respect to K, then the problem $NCP(f, K)$ has a solution.*

We also recall the following definitions. We say that the mapping $h : H \to H$ is *completely continuous*, if it is continuous and for any bounded set $B \subset H$, $h(B)$ is relatively compact. Let D be a nonempty subset of H. We say that a mapping $f\colon D \to H$ *satisfies condition* $(S)^1_+$ if any sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$ with (w)-$\lim_{n\to\infty} x_n = x_* \in H$, (w)-$\lim_{n\to\infty} f(x_n) = u \in H$ and

$$\limsup_{n\to\infty} \langle x_n, f(x_n)\rangle \le \langle x_*, u\rangle$$

has a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ convergent (in norm) to x_* ((w)-lim denotes the weak limit).

A mapping $f\colon H \to H$ is *scalarly compact* with respect to a closed convex set $D \subset H$, if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$ weakly convergent to an element $x_* \in D$ there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ such that $\limsup_{k\to\infty}\langle x_{n_k} - x_*, f(x_{n_k})\rangle \le 0$.

Finally, we recall that a mapping $f\colon H \to H$ is called *demicontinuous* if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset H$ convergent in norm to an element $x_* \in H$, $\{f(x_n)\}_{n\in\mathbb{N}}$ is weakly convergent to $f(x_*)$.

We proved in [10] the following result.

Theorem 3.1 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f\colon H \to H$ a mapping. If f has a decomposition of the form $f(x) = T_1(x) - T_2(x)$ such that:*

1. *T_1 is demicontinuous, bounded and satisfies condition $(S)^1_+$,*
2. *T_2 is demicontinuous and scalarly compact with respect to K,*

then f is REFE-acceptable with respect to K.

A long list of examples of REFE-acceptable mappings are given in [10]. Now, we add to that list another few examples of REFE-acceptable mappings as a consequence of Theorem 3.1.

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous mapping. We say that a mapping $T : H \to H$ is a ϕ-contraction if the following conditions are satisfied:

(i) $\|T(x) - T(y)\| \le \phi(\|x - y\|)$ for any $x, y \in H$.
(ii) $\phi(t) < t$ for any $t \in \mathbb{R}_+ \setminus \{0\}$.

We say that a mapping $T : H \to H$ is antimonotone if for any $x, y \in H$ we have $\langle x - y, f(x) - f(y)\rangle \le 0$.

Theorem 3.2 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone and $f\colon H \to H$ a mapping. If f has a decomposition of the form $f(x) = T_1(x) - T_2(x)$, where the mappings $T_1, T_2 : H \to H$ satisfy the following conditions:*

1. *T_1 is strongly monotone or $T_1 = I - T$, where $T : H \to H$ is a ϕ-contraction,*
2. *T_2 is antimonotone or $T_2 = h - g$, where h is completely continuous and g is monotone ($h, g : H \to H$),*
3. *T_1, T_2 are demicontinuous,*

then f is REFE-acceptable with respect to K.

Proof If T_1 satisfies one of the assumptions then T_1 satisfies condition $(S)^1_+$ (see [8] and [4]). If T_2 satisfies one of the assumptions then T_2 is scalarly compact (see [7]). Hence, the theorem is a consequence of Theorem 3.1. □

4 Mappings without REFE and existence theorems for complementarity problems

Definition 4.1 Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $f\colon H \to H$ a mapping. The *orthogonalizer* of f is the mapping $\mathcal{O}(f)\colon H \to H$ defined by

$$\mathcal{O}(f)(x) = \|x\|^2 f(x) - \langle f(x), x\rangle x.$$

It is easy to see that $\langle \mathcal{O}(f)(x), x\rangle = 0$, for all $x \in H$.

By Definition 4.1, Theorems 7.1 and 7.2 proved in [10] have the following reformulation:

Theorem 4.1 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone, $f\colon H \to H$ a mapping and $F = \mathcal{O}(f)$ the orthogonalizer of f. A necessary and sufficient condition for the mapping f to have the property of being without a REFE with respect to K is the following: There is a $\rho > 0$ such that for any $x \in K$ with $\|x\| = \rho$ at least one of the following conditions holds:*

1. $\langle f(x), x\rangle \geq 0$,
2. *x is not a solution of $NCP(F, K)$.*

If f is REFE-acceptable, then the problem $NCP(f, K)$ has a solution.

Proof It is enough to prove that condition 2 of Theorem 4.1 is equivalent to condition 2 of Theorem 7.1, [10]. Indeed, since $\langle F(x), x\rangle = 0$, for all $x \in H$, condition 2 of Theorem 4.1 is equivalent to $F(x) \notin K^*$, for any $x \in K$ with $\|x\| = \rho$, i.e., there is a $y \in K$ such that $\langle F(x), y\rangle < 0$, for any $x \in K$ with $\|x\| = \rho$. But this is exactly condition 2 of Theorem 7.1, [10]. □

Lemma 4.1 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $f\colon H \to H$ a mapping and $F = \mathcal{O}(f)$ its orthogonalizer. Then, $x \in H$ is an eigenvector of f if and only if $x \neq 0$ and $F(x) = 0$. The eigenvalue corresponding to an eigenvector $x \in H$ of f is*

$$\frac{\langle f(x), x\rangle}{\|x\|^2}.$$

Proof Suppose that x is an eigenvector of $f(x)$. Then, $x \neq 0$ and

$$f(x) = \lambda x, \tag{1}$$

where λ is the eigenvalue corresponding to the Eigenvector x. Multiplying (1) by x, we obtain

$$\lambda = \frac{\langle f(x), x\rangle}{\|x\|^2}.$$

Hence,

$$f(x) = \frac{\langle f(x), x\rangle}{\|x\|^2} x,$$

from which it follows that $F(x) = 0$. Conversely if $x \neq 0$ and $F(x) = 0$, then

$$f(x) = \frac{\langle f(x), x\rangle}{\|x\|^2} x,$$

and therefore x is an eigenvector of f with corresponding eigenvalue

$$\frac{\langle f(x), x\rangle}{\|x\|^2}.$$

□

Remark 4.1 We remark that by the proof of Theorem 4.1 the solutions of $NCP(F, K)$ coincide with the feasible points of $NCP(F, K)$. So in every result of the paper a solution of $NCP(F, K)$ could be replaced by a feasible point of $NCP(F, K)$.

Recall the following definition.

Definition 4.2 Let $H, \langle\cdot,\cdot\rangle$ be a Hilbert space and $K \subset H$ a closed convex cone. The subset U of the closed convex cone K is a *face of* K if it is a closed convex cone and if from $x \in U$, $y \in K$ and $x - y \in K$, it follows that $y \in U$. If "$\leq$" is the preorder induced by K, this can be written as: If $x \in U$ and $0 \leq y \leq x$, then $y \in U$.

Lemma 4.2 *Let* $H, \langle\cdot,\cdot\rangle$ *be a Hilbert space and* $K \subset H$ *a closed convex cone. If* U *is a face of* K *and* $U_\perp = \{z \in K^* \mid \langle x, z\rangle = 0$ *for all* $x \in U\}$*, then* $U_\perp$ *is a face of* K^*.

Proof By using the definition of a cone, it can be easily shown that $U_\perp$ is a cone. Denote by "$\leq_*$" the preorder induced by K^*, and let $z \in U_\perp$ and $t \in H$ such that $0 \leq_* t \leq_* z$. Then, $z - t \in K^*$ and $t \in K^*$. Hence, for all $x \in U \subset K$, we have $\langle x, z - t\rangle \geq 0$, or $0 \leq \langle x, t\rangle \leq \langle x, z\rangle = 0$. Thus, $\langle x, t\rangle = 0$, for all $x \in U$. Therefore, $t \in U_\perp$, proving that $U_\perp$ is a face of K^*. □

Definition 4.3 Let $H, \langle\cdot,\cdot\rangle$ be a Hilbert space and $K \subset H$ a closed convex cone. If U is a face of K, then $U_\perp = \{z \in K^* \mid \langle x, z\rangle = 0$ for all $x \in U\}$ is called the *orthogonal complement face to* U.

Lemma 4.3 *Let* $H, \langle\cdot,\cdot\rangle$ *be a Hilbert space,* $K \subset H$ *a closed convex cone and* $x \in K$*. Then the minimal face (with respect to set inclusion) of* K *containing* x *is* $V = \{y \in H : \exists\lambda \geq 0$ *such that* $0 \leq y \leq \lambda x\}$.

Proof First we prove that V is a face of K. By using the definition of a cone, it can be easily shown that V is a cone. Let $y \in V$ and $z \in H$ such that $0 \leq z \leq y$. Then, there is a $\lambda \geq 0$ such that $0 \leq z \leq y \leq \lambda x$. Hence, by the definition of V, $z \in V$. Therefore, V is a face of K. Let W be another face of K containing x. It remains to show that $V \subset W$. Consider an arbitrary point $y \in V$. Then, there is a $\lambda \geq 0$ such that $0 \leq y \leq \lambda x$. If $\lambda = 0$, then $y = 0 \in W$. Suppose now, that $\lambda \neq 0$. Since the preorder induced by a cone is invariant under the multiplications with positive scalars, the latter inequality implies that $0 \leq (1/\lambda)y \leq x$. Since $x \in W$ and W is a face of K, we obtain that $(1/\lambda)y \in W$. Since W is a cone, it follows that $y \in W$. We showed that in all cases $y \in V$ implies $y \in W$. Therefore, $V \subset W$. □

Lemma 4.4 *Let* $H, \langle\cdot,\cdot\rangle$ *be a Hilbert space,* $K \subset H$ *a closed convex cone,* $x \in K$ *and* V *the minimal face of* K *containing* x*. If* $y \in K^*$ *such that* $\langle x, y\rangle = 0$*, then* $y \in V_\perp$.

Proof We have to prove that $\langle y, z\rangle = 0$ for all $z \in V$. Since $z \in V$, there is a $\lambda \geq 0$ such that $0 \leq z \leq \lambda x$. Thus, $z \in K$ and $\lambda x - z \in K$. Since $y \in K^*$, it follows that $\langle y, \lambda x - z\rangle \geq 0$ and $0 = \lambda\langle y, x\rangle \geq \langle y, z\rangle \geq 0$. Therefore, $\langle y, z\rangle = 0$ for all $z \in V$. □

Theorem 4.2 *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $K \subset H$ *a closed convex cone,* $f: H \to H$ *a mapping and* $F = \mathcal{O}(f)$ *the orthogonalizer of* f*. Then, the mapping* f *is without a REFE with respect to* K *if and only if there is a* $\rho > 0$ *such that for any* $x \in K$ *with* $\|x\| = \rho$ *we have:*

1. *If* $x \in Int\,K$ *and* x *is an eigenvector of* f*, then its corresponding eigenvalue is nonnegative,*

2. *If* $x \in \partial K$ *and* $\langle f(x), x\rangle < 0$, *then* $F(x) \notin V_{\perp}$, *where* V *is the minimal face of* K *containing* x *and* $V_{\perp}$ *is the orthogonal complement face of* V *with respect to* K.

If f *is REFE-acceptable, then the problem* $NCP(f, K)$ *has a solution.*

Proof By Theorem 4.1 f is without a REFE if and only if there is a $\rho > 0$ such that for any $x \in K$ with $\|x\| = \rho$ at least one of the conditions 1 and 2 of Theorem 4.1 holds. Suppose that $x \in K$ and $\|x\| = \rho$. Let F be the orthogonalizer of f. We will show that if $x \in \operatorname{Int} K$ and $F(x) \neq 0$ (which by Lemma 4.1 is equivalent to x not being an eigenvector of f), then it follows that $F(x) \notin K^*$. Indeed, if we suppose that $F(x) \in K^*$, then from $x \in \operatorname{Int} K$ for any $y \in H$ with sufficiently small norm $x + y, x - y \in K$. Thus, from $\langle F(x), x + y\rangle \geq 0$ and $\langle F(x), x - y\rangle \geq 0$ it follows that $\langle F(x), y\rangle = 0$. Let $z \in H$ be an arbitrary vector. By multiplying with a sufficiently small nonzero constant λ we obtain $\langle F(x), \lambda z\rangle = 0$. Therefore, $\langle F(x), z\rangle = 0$, for any $z \in H$. By choosing $z = F(x)$, we obtain $F(x) = 0$ which is a contradiction. Therefore, $F(x) \notin K^*$, i.e., x is not a solution of $NCP(F, K)$. Hence, in this case condition 2 of Theorem 4.1 is satisfied. If $x \in \operatorname{Int} K$ and $F(x) = 0$, then x is a solution of $NCP(F, K)$. Hence, in this case condition 2 of Theorem 4.1 cannot hold. Moreover, by Lemma 4.1, x is an eigenvector of f with corresponding eigenvalue

$$\frac{\langle f(x), x\rangle}{\rho^2}.$$

Hence, in this case condition 1 of Theorem 4.1 holds if and only if x is an eigenvector of f with nonnegative eigenvalue.

Now, suppose that $x \in \partial K$. If $\langle f(x), x\rangle \geq 0$, then condition 1 of Theorem 4.1 holds. Suppose that $\langle f(x), x\rangle < 0$. Then, condition 1 of Theorem 4.1 cannot hold. Since $\langle F(x), x\rangle = 0$, Lemma 4.4 implies that $F(x) \in K^*$ is equivalent to $F(x) \in V_{\perp}$. Thus, condition 2 of Theorem 4.1 is equivalent to $F(x) \notin K^*$ which is equivalent to $F(x) \notin V_{\perp}$. □

Remark 4.2 Theorem 4.2 is true even if the interior of K is empty. In this case only condition 2 has to be considered. The same remark holds for any subsequent result in which condition 1 contains the interior of a cone.

Let $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be the n-dimensional euclidean space with the canonical scalar product and

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0, \ldots, x_n \geq 0\}$$

the positive orthant, which is a closed convex cone. The interior of $\mathbb{R}^n_+$ is

$$\mathbb{R}^n_{++} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > 0, \ldots, x_n > 0\}$$

The boundary of $\mathbb{R}^n_+$ is

$$\partial\mathbb{R}^n_+ = \mathbb{R}^n_+ \setminus \mathbb{R}^n_{++} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+ \mid x_i = 0,\ \text{for some } i \in \{1, \ldots, n\}\}$$

Corollary 4.1 *Let* $f\colon \mathbb{R}^n \to \mathbb{R}^n$ *be a mapping and* $F = \mathcal{O}(f)$ *the orthogonalizer of* f. *Then, the mapping* f *is without a REFE with respect to the positive orthant* $\mathbb{R}^n_+$ *if and only if there is a* $\rho > 0$ *such that for any* $x \in \mathbb{R}^n_+$ *with* $\|x\| = \rho$ *we have:*

1. *if* $x \in \mathbb{R}^n_{++}$ *and* x *is an eigenvector of* f, *then its corresponding eigenvalue is nonnegative,*
2. *if* $x \in \partial\mathbb{R}^n_+$ *and* $\langle f(x), x\rangle < 0$, *then* $F(x) \notin V_{\perp}$, *where* V *is the minimal face of* $\mathbb{R}^n_+$ *containing* x *and* $V_{\perp}$ *is the orthogonal complement face of* V *with respect to* $\mathbb{R}^n_+$.

By Lemma 4.1, [10] and Corollary 4.1, we have:

Corollary 4.2 *Let* $f\colon \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous mapping and* $F = \mathcal{O}(f)$ *the orthogonalizer of* f*. If there is a* $\rho > 0$ *such that for any* $x \in \mathbb{R}^n_+$ *with* $\|x\| = \rho$ *we have:*

1. *if* $x \in \mathbb{R}^n_{++}$ *and* x *is an eigenvector of* f*, then its corresponding eigenvalue is nonnegative,*
2. *if* $x \in \partial\mathbb{R}^n_+$ *and* $\langle f(x), x\rangle < 0$*, then* $F(x) \notin V_\perp$*, where* V *is the minimal face of* $\mathbb{R}^n_+$ *containing* x *and* $V_\perp$ *is the orthogonal complement face of* V *with respect to* $\mathbb{R}^n_+$*,*

then the problem $NCP(f, \mathbb{R}^n_+)$ *has a solution.*

Corollary 4.3 *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space,* $K \subset H$ *a closed convex cone, and* $f\colon H \to H$ *a REFE-acceptable mapping. If there is a* $\rho > 0$ *such that for any* $x \in K$ *with* $\|x\| = \rho$ *we have:*

1. *If* $x \in \operatorname{Int} K$ *and* x *is an eigenvector of* f*, then its corresponding eigenvalue is nonnegative,*
2. *If* $x \in \partial K$ *and* $\langle f(x), x\rangle < 0$*, then* $f(x) \notin V_\perp - V$*, where* V *is the minimal face of* K *containing* x *and* $V_\perp \subset K^*$ *is the orthogonal complement face of* V *with respect to* K*,*

then the problem $NCP(f, K)$ *has a solution. Particularly, if* $H = \mathbb{R}^n$ *and* $K = \mathbb{R}^n_+$*, then conditions 1 and 2 become*

1′. *If* $x \in \mathbb{R}^n_{++}$ *and* x *is an eigenvector of* f*, then its corresponding eigenvalue is nonnegative,*
2′. *If* $x \in \partial\mathbb{R}^n_+$ *and* $\langle f(x), x\rangle < 0$*, then* $f(x) \notin V_\perp - V$*, where* V *is the minimal face of* $\mathbb{R}^n_+$ *containing* x *and* $V_\perp$ *is the orthogonal complement face of* V *with respect to* $\mathbb{R}^n_+$*.*

Proof Let $F = \mathcal{O}(f)$ be the orthogonalizer of f. We shall use Theorem 4.2. Condition 1 of Corollary 4.3 coincides with condition 1 of Theorem 4.2. Hence, it is enough to prove that condition 2 of Corollary 4.3 implies condition 2 of Theorem 4.2. Suppose that condition 2 of Corollary 4.3 is satisfied and let $x \in \partial K$ with $\|x\| = \rho$ such that $\langle f(x), x\rangle < 0$. Then, $f(x) \notin V_\perp - V$, where V is the minimal face of K containing x and $V_\perp$ is the orthogonal complement face of V with respect to K. We have to prove that $F(x) \notin V_\perp$. Indeed, if we suppose to the contrary that $F(x) \in V_\perp$, then $\rho^2 f(x) - \langle f(x), x\rangle x \in V_\perp$, which by using relation $\langle f(x), x\rangle < 0$ implies that $f(x) \in V_\perp - V$ (since the cones V and $V_\perp$ are invariant under the multiplication by positive scalars). But this is a contradiction. Therefore, $F(x) \notin V_\perp$. □

Remark 4.3 With the notations of Corollary 4.3 Condition 2′, if $x \neq 0$ and $f(x) \in V_\perp - V$, then $\langle f(x), x\rangle < 0$. Indeed, suppose that $f(x) = y - z$, where $y \in V_\perp$ and $z \in V$. Then, $\langle f(x), x\rangle = \langle y - z, x\rangle = \langle y, x\rangle - \langle z, x\rangle = -\langle z, x\rangle$. Since V is the minimal face containing x, x is in the relative interior of V. Hence, $\langle z, x\rangle > 0$. Therefore, $\langle f(x), x\rangle < 0$.

For any $x \in \mathbb{R}^n$ let

$$I_-(x) = \{i \in \{1, 2, \ldots, n\} \mid x_i < 0\},$$
$$I_0(x) = \{i \in \{1, 2, \ldots, n\} \mid x_i = 0\},$$
$$I_+(x) = \{i \in \{1, 2, \ldots, n\} \mid x_i > 0\},$$

where $x = (x_1, \ldots, x_n)$. Then, Corollary 4.3 has the following consequence:

Corollary 4.4 *Let* $f\colon \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous mapping. If there is a* $\rho > 0$ *such that for any* $x \in \mathbb{R}^n_+$ *with* $\|x\| = \rho$ *we have:*

1. *If* $x \in \mathbb{R}^n_{++}$ *and* x *is an eigenvector of* f*, then its corresponding eigenvalue is nonnegative,*
2. *If* $x \in \partial\mathbb{R}^n_+$ *and* $\langle f(x), x\rangle < 0$*, then either* $I_0(x) \neq I_+(f(x))$ *or* $I_+(x) \neq I_-(f(x))$*,*

then the problem $NCP(f, \mathbb{R}^n_+)$ *has a solution.*

Proof We shall use Corollary 4.3. It is enough to prove that condition 2 of Corollary 4.4 implies condition 2′ of Corollary 4.3. It is easy to see that $y \in V_\perp$ if and only if for any $x \in V$, we have $I_0(y) = I_+(x)$, where V and $V_\perp$ are defined in Corollary 4.3. Of course, this last relation is equivalent to $I_+(y) = I_0(x)$. Hence, if $f(x) \in V_\perp - V$, then $I_0(x) = I_+(f(x))$ and $I_+(x) = I_-(f(x))$. Thus, if either $I_0(x) \neq I_+(f(x))$ or $I_+(x) \neq I_-(f(x))$, then $f(x) \notin V_\perp - V$. Therefore condition 2 of Corollary 4.4 implies condition 2′ of Corollary 4.3. □

Corollary 4.4 can be written more explicitly as follows:

Corollary 4.5 *Let* $f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuous mapping. If there is a* $\rho > 0$ *such that for any* $x \in \mathbb{R}^n_+$ *with* $\|x\| = \rho$ *we have:*

1. *If* $x \in \mathbb{R}^n_{++}$ *and* x *is an eigenvector of* f*, then its corresponding eigenvalue is nonnegative,*
2. *If* $x = (x_1, \ldots, x_n) \in \partial\mathbb{R}^n_+$ *and* $\langle f(x), x\rangle < 0$*, then* $\exists i_0 \in \{1, \ldots, n\}$ *such that*

$$x_{i_0} = 0 \ \wedge \ f_{i_0}(x) \leq 0 \ \vee \ x_{i_0} > 0 \ \wedge \ f_{i_0}(x) \geq 0,$$

where $\wedge$ *and* $\vee$ *denotes the "logical and" and "logical or", respectively, then the problem* $NCP(f, \mathbb{R}^n_+)$ *has a solution.*

Proof Condition 2 of Corollary 4.4 is equivalent to

$$\begin{aligned} x_{i_0} = 0 \ \wedge \ f_{i_0}(x) \leq 0 \ \vee \ x_{i_0} > 0 \ \wedge \ f_{i_0}(x) > 0 \\ \vee \ x_{i_0} > 0 \ \wedge \ f_{i_0}(x) \geq 0 \ \vee \ x_{i_0} = 0 \ \wedge \ f_{i_0}(x) < 0, \end{aligned}$$

which by using the associativity of $\vee$ and the distributivity of the $\wedge$ with respect to $\vee$ can be written as

$$x_{i_0} = 0 \wedge (f_{i_0}(x) \leq 0 \ \vee \ f_{i_0}(x) < 0) \ \vee \ x_{i_0} > 0 \ \wedge \ (f_{i_0}(x) > 0 \ \vee \ f_{i_0}(x) \geq 0),$$

or by simplification

$$x_{i_0} = 0 \ \wedge \ f_{i_0}(x) \leq 0 \ \vee \ x_{i_0} > 0 \ \wedge \ f_{i_0}(x) \geq 0.$$

□

Denote by I the identity operator of $\mathbb{R}^n$, i.e., $I(x) = x$, for all $x \in \mathbb{R}^n$.

Lemma 4.5 *If* $f(x) = Ax + b$*, where* $A : \mathbb{R}^n \to \mathbb{R}^n$ *is a linear mapping and* b *is a nonzero constant vector, then* λ *is an eigenvalue of* f *if and only if is not an eigenvalue of* A*. If* λ *is an eigenvalue of* f*, then* $x = (A - \lambda I)^{-1}b$ *is the only eigenvector corresponding to* λ*.*

Proof If λ is an eigenvalue of f and x is an eigenvector corresponding to λ, then $Ax+b = \lambda x$, i.e.,

$$(A - \lambda I)x = b. \tag{2}$$

This linear system of equations has a solution if and only if $\det(A - \lambda I) \neq 0$, i.e., λ is not an eigenvalue of A. In this case equation (2) implies that $x = (A - \lambda I)^{-1} b$ is the only eigenvector corresponding to λ. □

By Lemma 4.5 and Corollary 4.5 we have as follows:

Corollary 4.6 *Let $f(x) = Ax + b$, where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping with entries a_{ij}, $i, j \in \{1, \dots, n\}$ with respect to the canonical basis of $\mathbb{R}^n$ and $b = (b_1, \dots, b_n)$ is a nonzero constant vector. If there is a $\rho > 0$ such that for any $x \in \mathbb{R}^n_+$ with $\|x\| = \rho$ we have:*

1. *If λ is not an eigenvalue of A and $x = (A - \lambda I)^{-1} b \in \mathbb{R}^n_{++}$, then λ is nonnegative,*
2. *If $x = (x_1, \dots, x_n) \in \partial\mathbb{R}^n_+$ and $\sum_{i,j=1}^n a_{ij} x_i x_j + \sum_{i=1}^n b_i x_i < 0$, then $\exists i_0 \in \{1, \dots, n\}$ such that*

$$x_{i_0} = 0 \ \wedge \ \sum_{j=1}^n a_{i_0 j} x_j + b_{i_0} \leq 0 \ \vee \ x_{i_0} > 0 \ \wedge \ \sum_{j=1}^n a_{i_0 j} x_j + b_{i_0} \geq 0 \ ,$$

then the linear complementarity problem $LCP(A, b, \mathbb{R}^n_+)$ has a solution.

From a result proved in [9], in the case of pseudomonotone REFE-accep-table mappings the nonexistence of REFE is equivalent to the solvability of the corresponding nonlinear complementarity problem. Therefore, by Lemma 4.1, [10] and Theorem 4.2 we have as follows:

Theorem 4.3 *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed convex cone, Let $f : H \to H$ be a REFE-acceptable pseudomonotone mapping and $F = \mathcal{O}(f)$ the orthogonalizer of f. Then, the problem $NCP(f, K)$ has a solution if and only if there is a $\rho > 0$ such that for any $x \in K$ with $\|x\| = \rho$ we have:*

1. *If $x \in \operatorname{Int} K$ and x is an eigenvector of f, then its corresponding eigenvalue is nonnegative,*
2. *If $x \in \partial K$ and $\langle f(x), x \rangle < 0$, then $F(x) \notin V_\perp$, where V is the minimal face of K containing x and $V_\perp \subset K^*$ is the orthogonal complement face of V with respect to K.*

Corollary 4.7 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous pseudomonotone mapping and $F = \mathcal{O}(f)$ the orthogonalizer of f. Then, the problem $NCP(f, \mathbb{R}^n_+)$ has a solution if and only if there is a $\rho > 0$ such that for any $x \in \mathbb{R}^n_+$ with $\|x\| = \rho$ we have:*

1. *If $x \in \mathbb{R}^n_{++}$ and x is an eigenvector of f, then its corresponding eigenvalue is nonnegative,*
2. *If $x \in \partial\mathbb{R}^n_+$ and $\langle f(x), x \rangle < 0$, then $F(x) \notin V_\perp$, where V is the minimal face of $\mathbb{R}^n_+$ containing x and $V_\perp$ is the orthogonal complement face of V with respect to $\mathbb{R}^n_+$.*

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